Boundary Value Problems of Thermoelasticity for Porous Sphere And for a Space with Spherical Cavity

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Abstract

The present paper is devoted to construct explicit solutions of the quasi-static boundary value problems (BVPs) of coupled theory of thermoelasticity for a porous elastic sphere and for a space with a spherical cavity. In this research the regular solution of the system of equations for an isotropic porous material is constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. The basic BVPs for a sphere and for a space with a spherical cavity are solved explicitly. The obtained solutions are given by means of the harmonic, bi-harmonic and meta-harmonic functions. For the harmonic functions the Poisson type formulas are obtained. The bi-harmonic and meta-harmonic functions are presented as absolutely and uniformly convergent series.

1.Introduction

In most solids there are pores through which the liquid or gas may flow. Many materials such as rocks, sand, soil are known as porous materials, the human skin has a larger number of pores, cancellous bone is considered as a porous material and etc.

The foundations of the theory of elastic materials with voids were first proposed by Cowin and Nunziato [1,2]. They investigated the linear and nonlinear theories of elastic materials with voids. In these theories the independent variables are displacement vector field and the change of volume fraction of pores. Such materials include, rocks and soils, granulated and some other manufactured porous materials. The history of development of porous body mechanics, the main results and the sphere of their application are set forth in detail in the monographs [3-6] (see references therein). The generalization of the theory of elasticity and thermoelasticity for materials with double voids belongs to Ieşan and Quintanilla [7]. In [8] Svanadze considered the coupled linear model of porous elastic solids by combining the following three variables: the displacement vector field, the volume fraction of pores and the change of fluid pressure in pore network. In this work the basic internal and external BVPs of steady vibrations are investigated, the uniqueness and the existence theorems are proved by means of the potential method and the theory of singular integral equations. The coupled linear theory of thermoelasticity for isotropic porous materials by using the concept of Darcy's law and the volume fraction of pore network are presented by Svanadze [9]. The quasi-static BVPs of the theories of elasticity and thermoelasticity for porous materials are studied by Mikelashvili [10,11].

For applications, it is especially important to construct the solutions of BVPs in explicit form because such solutions enable us to effectively perform quantitative analysis of the investigated problems. Questions related to this topic, different types of problems in the theory of elasticity and thermoelasticity of porous materials are considered, for example, in the works [12-25], where the explicit solutions are constructed for some BVPs for the concrete domains.

The present paper is devoted to construct explicit solutions of the quasi-static boundary value problems (BVPs) of coupled theory of thermoelasticity for porous elastic sphere and for a space with a spherical cavity. The regular solution of the system of equations for an isotropic porous material is constructed by means of the elementary (harmonic, bi-harmonic and meta-harmonic) functions. The basic BVPs for a sphere and for a space with a spherical cavity are solved explicitly.

2.Basic equations and boundary value problems

Let $x=(x_1,x_2,x_3)$ be a point in the Euclidean three-dimensional space E^3. We consider an isotropic thermoelastic porous sphere D bounded by a surface S with center at the origin and radius R. Let D^- be the whole space with a spherical cavity and with boundary S.

The basic homogeneous system of equations in the coupled linear quasi-static theory of thermoelasticity for porous materials expressed in terms of the displacement vector $\mathbf{u}(\mathbf{x})$, the change of the volume fraction φ, the change of fluid pressure in pore network p and the variation of temperature θ has the following form [11]:

$$\begin{aligned}
&\mu\,\Delta\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}+\operatorname{grad}(b\varphi-\beta\,p-\gamma_0\theta)=0,\\
&(\alpha\,\Delta-\alpha_1)\varphi-b\operatorname{div}\mathbf{u}+mp+\gamma_1\theta=0,\\
&(k\,\Delta+i\omega\,a_0)\,p+i\omega(\beta\operatorname{div}\mathbf{u}+m\,\varphi+\gamma_2\,\theta)=0,\\
&(k_0\,\Delta+i\omega\,aT_0)\theta+i\omega T_0(\gamma_0\operatorname{div}\mathbf{u}+\gamma_1\,\varphi+\gamma_2\,p)=0,
\end{aligned}\qquad(1)$$

where γ_0 is the thermal expansion coefficient, a_0 is measures the compressibility of pores, $\alpha,\ \alpha_1,\ b,\ m,\ \gamma_1,\gamma_2$ are the constitutive coefficients, $k=\dfrac{k'}{\mu'}$, μ' is the fluid viscosity, k' is the macroscopic intrinsic permeability associated with the pore network , k_0 is the thermal conductivity of the porous material, a is the heat capacity, λ, μ are the Lame constants, β is the effective stress parameter, ω is the oscillation frequency, $\omega > 0$, Δ is the 3D Laplace operator.

Definition 1. *A vector-function* $\mathrm{U} = (\mathbf{u}, \varphi, p, \theta)^T$ *defined in the domain* $D(D^-)$ *is called regular if*

$$\mathbf{U}(\mathbf{x}) \in C^2(D) \cap C^1(\overline{D}) \quad \left(\mathbf{U}(\mathbf{x}) \in C^2(D^-) \cap C^1(\overline{D^-})\right)$$ *and the following conditions at infinity are added:*

$$\mathbf{U} \in O(|\mathbf{x}|^{-1}), \qquad \frac{\partial \mathbf{U}}{\partial x_j} \in O(|\mathbf{x}|^{-2}),$$

$$j = 1,2,3, \quad |\mathbf{x}|^2 = x_1^2 + x_2^2 + x_3^2.$$

For the system (1), we pose the following boundary value problems:

Problem 1: (The Dirichlet type BVP) Find a regular solution $\mathbf{U}$, satisfying in D the system of equations (1), if on the boundary S the following conditions are given:

$$\mathbf{u}^+ = \mathbf{f}^+(\mathbf{z}), \quad \varphi^+ = f_3^+(\mathbf{z}), \quad p^+ = f_4^+(\mathbf{z}),$$
$$\theta^+ = f_5, \quad z \in S,$$

Problem 2: (The Neumann type BVP)Find a regular solution $\mathbf{U}$, satisfying in D^- the system of equations (1), if on the boundary S the following conditions are given:

$$(\mathbf{P}(\partial_x, \mathbf{n})\mathbf{u})^- = \mathbf{f}^-(\mathbf{z}), \quad \left(\frac{\partial \varphi}{\partial n}\right)^- = f_3^-(\mathbf{z}),$$

$$\left(\frac{\partial p}{\partial n}\right)^- = f_4^-(\mathbf{z}), \quad \left(\frac{\partial \theta}{\partial n}\right)^{--} = f_5, \quad z \in S,$$

where $\mathbf{f}^\pm(f_1, f_2, f_3)$, $f_4^\pm$, $f_5^\pm$ are the given functions. Moreover, we assume that $f_i^\pm$ can be presented in the form of series, $(\cdot)^\pm$ denotes the limiting value from $D^\pm$,

$$\mathbf{U}^+(\mathbf{z}) = \lim_{D^+ \supset x \to z \in S} \mathbf{U}(\mathbf{x}), \quad \mathbf{U}^-(\mathbf{z}) = \lim_{D^- \supset x \to z \in S} \mathbf{U}(\mathbf{x}),$$

the vector $\mathbf{P}(\partial_x, \mathbf{n})\mathbf{u}$ is defined in the following form

$$\mathbf{P}(\partial_x, \mathbf{n})\mathbf{u} = \mathbf{T}(\partial_x, \mathbf{n})\mathbf{u} + \mathbf{n}(b\varphi - \beta p - \gamma_0 \theta),$$

$\mathbf{T}(\partial_x, \mathbf{n})\mathbf{u}$ is the stress vector in the classical theory of elasticity

$$\mathbf{T}(\partial_x, \mathbf{n})\mathbf{u} = 2\mu \frac{\partial \mathbf{u}}{\partial n} + \lambda \mathbf{n} \operatorname{div} \mathbf{u} + \mu[\mathbf{n} \cdot \operatorname{rot} \mathbf{u}],$$

$\mathbf{n}$ is the external normal vector on S at $\mathbf{z} \in S$.

The following assertion holds (for details see [11]).

Theorem 1. *The Problem 1(Problem 2) has one regular solution in* $D(D^-)$.

The purpose of this paper is to construct an explicit solution of system (1) for a porous sphere and for a porous space with a spherical cavity.

3.A representation of regular solution

The following theorems holds:

Theorem 2. *If* $\mathbf{U}$ *is a regular solution of the system (1) then the functions* $\mathbf{u}$, $\operatorname{div} \mathbf{u}$, φ, p *and* θ *satisfy the following equations:*

$$\begin{aligned} &\Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\mathbf{u} = 0, \\ &\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\mathbf{\Phi} = 0, \end{aligned} \tag{2}$$

where λ_j^2, $j = 1,2,3,$ are roots of the third-degree algebraic equation(see below), $\mathbf{\Phi} = (\operatorname{div} \mathbf{u}, \varphi, p, \theta)$.

Proof. Let $\mathbf{U}$ be a regular solution of the equation (1). Applying the divergence operator to Eq. $(1)_1$, we obtain

$$\begin{aligned} &\mu_0 \Delta \operatorname{div} \mathbf{u} + \Delta(b\varphi - \beta p - \gamma_0 \theta) = 0, \\ &(\alpha \Delta - \alpha_1)\varphi - b \operatorname{div} \mathbf{u} + mp + \gamma_1 \theta = 0, \\ &(k\Delta + i\omega a_0)p + i\omega(\beta \operatorname{div} \mathbf{u} + m\varphi + \gamma_2 \theta) = 0, \\ &(k_0 \Delta + i\omega a T_0)\theta + i\omega T(\gamma_0 \operatorname{div} \mathbf{u} + \gamma_1 \varphi + \gamma_2 p) = 0. \end{aligned} \tag{3}$$

Let us rewrite the system (3) as follows

$$D(\Delta)\mathbf{\Phi} = \begin{pmatrix} \mu_0\Delta & b\Delta & -\beta\Delta & -\gamma_0\Delta \\ -b & \alpha\Delta - \alpha_1 & m & \gamma_1 \\ i\omega\beta & i\omega m & k\Delta + i\omega a_0 & i\omega\gamma_2 \\ i\omega T_0\gamma_0 & i\omega T_0\gamma_1 & i\omega T_0\gamma_2 & k_0\Delta + i\omega a T_0 \end{pmatrix}\mathbf{\Phi} = 0.$$

As we will see

$$\det D(\Delta) = \alpha\, \mu_0 k_0 k \Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2),$$

where λ_j^2, $j = 1,2,3$ are roots of third-degree algebraic equation with respect to ξ

$$\alpha\, \mu_0 k_0 k \xi^3 - d_1 \xi^2 + i\omega d_2 \xi - (i\omega)^2 T_0 d_3 = 0,$$
$$\mu_0 = \lambda + 2\mu,$$

$$\begin{aligned} d_1 = {} & \mu_0\left[-\alpha_1 k k_0 + i\omega\alpha(akT_0 + a_0 k_0)\right] \\ & + i\omega\alpha(k_0\beta^2 + k\gamma_0^2 T_0) + b^2 k k_0, \end{aligned}$$

$$d_2 = -\mu_0 k_0(\alpha_1 a_0 + m^2) - i\omega\alpha T_0\mu_0(\gamma_2^2 - aa_0)$$
$$-\mu_0 T_0 k(a\alpha_1 + \gamma_1^2) +$$
$$+k_0(a_0 b^2 - \alpha_1\beta^2 + 2bm\beta) + T_0 k(ab^2 + 2\gamma_0\gamma_1 b - \gamma_0^2\alpha_1)$$
$$+T_0 i\omega\alpha(a\beta^2 - 2\gamma_0\gamma_2\beta + a_0\gamma_0^2),$$
$$d_3 = \mu_0(\alpha_1\gamma_2^2 - a_0\gamma_1^2 + 2m\gamma_1\gamma_2) - \mu_0 a(\alpha_1 a_0 + m^2)$$
$$+a\left[a_0 b^2 - \alpha_1\beta^2 + 2bm\beta\right] + 2\gamma_0\gamma_1(a_0 b + 2m\beta)$$
$$+2\alpha_1\beta\gamma_0\gamma_2 - a_0\alpha_1\gamma_0^2 - (m\gamma_0 + b\gamma_2 + \beta\gamma_1)^2.$$

We assume that $\lambda_j^2, \quad j = 1,2,3$ are distinct and different from zero. We may assume without loss of generality that $\operatorname{Im}\lambda_j^2 > 0, \quad j = 1,2,3.$ [11]

It is obvious that, from relation

$$D(\Delta)\mathbf{\Phi} = 0$$

follows the following equations

$$\begin{aligned}
&\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\,\mathrm{div}\,\mathbf{u} = 0,\\
&\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\varphi = 0,\\
&\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)p = 0,\\
&\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\theta = 0.
\end{aligned} \tag{4}$$

Further, applying the operator $\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)$ to equation $(1)_1$ and using relations (4), we obtain

$$\Delta\Delta(\Delta + \lambda_1^2)(\Delta + \lambda_2^2)(\Delta + \lambda_3^2)\mathbf{u} = 0. \tag{5}$$

The prove is done.

Theorem 3. *The regular solution* $\mathbf{U}(\mathbf{u}, \varphi, p, \theta)$ *of the system (1) can be represented as follows:*

$$\begin{aligned}
&\mathbf{u} = \mathbf{\Psi} - \mathrm{grad}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right],\\
&\varphi = A_0 h + \sum_{j=1}^{3} A h_j, \quad p = B_0 h + \sum_{j=1}^{3} B_j h_j,\\
&\theta = C_0 h + \sum_{j=1}^{3} C_j h_j,
\end{aligned} \tag{6}$$

where

$$\mathrm{div}\,\mathbf{u} = h + \sum_{j-1}^{3} h_j, \quad \mathrm{div}\,\mathbf{\Psi} = m_0 h, \quad \Delta h_0 = h,$$
$$\Delta h = 0, \quad (\Delta + \lambda_j^2)h_j = 0,$$

$$A_0\delta_0 = a(a_0 b + m\beta) - b\gamma_2^2 - m\gamma_0\gamma_2$$
$$-\beta\gamma_2\gamma_1 + a_0\gamma_0\gamma_1,$$
$$B_0\delta_0 = a(\alpha_1\beta - bm) + \beta\gamma_1^2 - \alpha_1\gamma_0\gamma_2$$
$$+b\gamma_2\gamma_1 - m\gamma_0\gamma_1,$$
$$C_0\delta_0 = \gamma_0(\alpha_1 a_0 + m^2) - \gamma_1(m\beta + ba_0)$$
$$+\gamma_2(mb - \alpha_1\beta), \quad m_0 = \frac{d_3}{\mu\delta_0},$$

$$\begin{aligned}
&\delta_j = -\alpha k k_0\lambda_j^6 +\\
&\lambda_j^4\left[-\alpha_1 k k_0 + i\omega\alpha(akT_0 + a_0 k_0)\right] + (i\omega)^2 T_0\delta_0 +\\
&\lambda_j^2 i\omega\left[(a_0\alpha_1 + m^2)k_0 + T_0\, k(\gamma_1^2 + a\alpha_1)\right] +\\
&\lambda_j^2(i\omega)^2\alpha T_0(\gamma_2^2 - \alpha\, a_0),
\end{aligned} \tag{7}$$

$$\delta_0 = -a(\alpha_1 a_0 + m^2) + \alpha_1\gamma_2^2 - a_0\gamma_2^2 + 2m\gamma_1\gamma_2,$$

$$A_j\delta_j = bkk_0\lambda_j^4 - \lambda_j^2 i\omega\left[k_0(ba_0 + \beta m) + kT_0(\gamma_0\gamma_1 + ab\right]$$
$$+(i\omega)^2 T_0 A_0\delta_0, \quad j = 1,2,3,$$

$$B_j\delta_j = -i\omega\alpha\beta\, k_0\lambda_j^4 + (i\omega)^2 T_0 B_0\delta_0 -$$
$$\lambda_j^2 i\omega\left[-i\omega\alpha T_0(\gamma_0\gamma_2 - \alpha\beta) - k_0(\beta\alpha_1 - bm)\right],$$

$$C_j\delta_j = -i\omega\alpha T_0\gamma_0 k\lambda_j^4 + (i\omega)^2 T_0 C_0\delta_0$$
$$+\lambda_j^2 i\omega T_0\left[-i\omega\alpha(\beta\gamma_2 - a_0\gamma_0) + k(-\alpha_1\gamma_0 + b\gamma_1)\right],$$

$$\mu_0 + bA_0 - \beta B_0 - \gamma_0 C_0 = \frac{d_3}{\delta_0} = \mu\, m_0,$$

$$bA_j - \beta B_j - \gamma_0 C_j = -\mu_0.$$

Proof. It is well known, that the general solutions $\mathrm{div}\,\mathbf{u}$, φ, p and θ of equations (4) can be written as follows [26]

$$\begin{aligned}
&\mathrm{div}\,\mathbf{u} = h + \sum_{j-1}^{2} h_j, \quad \varphi = A_0 h + \sum_{j=1}^{2} A_j h_j,\\
&p = B_0 h + \sum_{j=1}^{2} B_j h_j, \quad \theta = C_0 h + \sum_{j=1}^{2} C_j h_j.
\end{aligned} \tag{8}$$

It can be easily checked that, they are the solutions of Eqs. $(1)_2, (1)_3, (1)_4$. If supposing that h and h_j are known functions, taking into account the representations (8), from Eq. $(1)_1$ to define the displacement vector $\mathbf{u}$ we get the following non-homogeneous equation

$$\Delta\mathbf{u} = -\mathrm{grad}\left[(m_0 - 1)h - \sum_{j=1}^{3} h_j\right], \tag{9}$$

solution of which can be written as

$$\mathbf{u} = \mathbf{\Psi} + \mathbf{u}_0,$$

where $\mathbf{\Psi}$ is an arbitrary harmonic function, $\mathbf{u}_0$

is a particular solution of equation (9)

$$\mathbf{u}_0 = -\,\mathrm{grad}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right]. \qquad (10)$$

Herein it is assumed that, the functions $\operatorname{div}\mathbf{u}$ $\mathbf{\Psi}$, h_j and h are interrelated by the following relations

$$\operatorname{div}\mathbf{u} = h + \sum_{j-1}^{3} h_j, \quad \operatorname{div}\mathbf{\Psi} = m_0 h,$$

$$\Delta h_0 = h, \quad \Delta h = 0, \quad \Delta\mathbf{\Psi} = 0.$$

Thus, From the above reasoning we have obtained the general solution of the system (1)in the following form

$$\mathbf{u} = \mathbf{\Psi} - \mathrm{grad}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right],$$

$$\varphi = A_0 h + \sum_{j=1}^{3} A_j h_j, \quad p = B_0 h + \sum_{j=1}^{3} B_j h_j,$$

$$\theta = C_0 h + \sum_{j=1}^{3} C_j h_j, \qquad (11)$$

where A_j, $B_j, C_j, j = 0,1,2,3$ and m_0 are given by (7).

From (11) we conclude that the representation of a solution of $\mathbf{u}$ contains a harmonic, bi-harmonic, and a meta-harmonic functions, while the representations of φ, p and θ contains only a harmonic and a meta-harmonic functions.

4.Explicit Solution of the Problem 1

Let us introduce the spherical coordinates equalities

$$x_1 = \rho\sin\xi\cos\eta, \quad x_2 = \rho\sin\xi\sin\eta, \quad x_3 = \rho\cos\xi,$$

$$\rho = \sqrt{x_1^2 + x_2^2 + x_3^2}, \quad 0 \le \eta \le 2\pi.$$

$$y_1 = R\sin\xi_0\cos\eta_0, \quad y_2 = R\sin\xi_0\sin\eta_0,$$

$$y_3 = R\cos\xi_0.$$

Taking into account the identity $(\mathbf{x}\cdot\mathrm{grad}) = \rho\dfrac{\partial}{\partial\rho}$, by direct calculation, from (11) we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = (\mathbf{x}\cdot\mathbf{\Psi}) - \rho\frac{\partial}{\partial\rho}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right]. \qquad (12)$$

It is easily to verified, that the function $(\mathbf{x}\cdot\mathbf{\Psi})$ satisfies the following equation

$$\Delta(\mathbf{x}\cdot\mathbf{\Psi}) = 2\operatorname{div}\mathbf{\Psi} = 2m_0 h, \qquad (13)$$

the solution of which has the form

$$(\mathbf{x}\cdot\mathbf{\Psi}) = \Omega + 2m_0 h_0,$$

where Ω is an arbitrary harmonic function $\Delta\Omega = 0,$ the function h_0 is a bi-harmonic function and chosen such that $\Delta h_0 = h, \quad \Delta h = 0.$

Substituting the expression (13) into (12) and taking into account (11), we obtain

$$(\mathbf{x}\cdot\mathbf{u}) = \Omega + 2m_0 h_0 - \rho\frac{\partial}{\partial\rho}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right],$$

$$\varphi = A_0 h + \sum_{j=1}^{3} A_j h_j, \quad p = B_0 h + \sum_{j=1}^{3} B_j h_j, \qquad (14)$$

$$\theta = C_0 h + \sum_{j=1}^{3} C_j h_j, \quad \operatorname{div}\mathbf{u} = h + \sum_{j=1}^{3} h_j.$$

We are looking for a solution of the system (1), under BCs of Problem 1, in the form (11), where the functions h, h_j, and Ω are sought in the form [26]

$$h = \sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n Z_n(\xi,\eta), \qquad \Omega = \sum_{n=0}^{\infty}\left(\frac{\rho}{R}\right)^n Y_n(\xi,\eta),$$

$$h_j = \sum_{n=0}^{\infty}\Phi_n(\lambda_j\rho)Y_{jn}(\xi,\eta), \quad j = 1,2,3, \quad \rho < R,$$

(15)

Z_n, Y_n, and Y_{jn}, $j = 1,2,3,$ are the unknown spherical harmonics of order n,

$$\Phi_n(\lambda_j\rho) = \frac{\sqrt{R}J_{n+\frac{1}{2}}(\lambda_j\rho)}{\sqrt{\rho}J_{n+\frac{1}{2}}(\lambda_j R)},$$

$J_{n+\frac{1}{2}}(\lambda_j\rho)$ is the Bessel function.

Taking into account (15), we can write the particular solutions of equation $\Delta h_0 = h$ in the form

$$h_0 = \frac{\rho^2}{2}\sum_{n=0}^{\infty}\frac{1}{3+2n}\left(\frac{\rho}{R}\right)^n Z_n(\theta,\eta). \qquad (16)$$

For convenience we introduce the following functions:

$$(\mathbf{x}\cdot\mathbf{f})^+ = g_1^+, \quad (\operatorname{div}\mathbf{f})^+ = g_2^+,$$
$$\varphi^+ = g_3^+, \quad p^+ = g_4^+, \quad \theta = g_5^+. \qquad (17)$$

We assume that the functions g_k^+ can be represented in the form of series

$$g_k^+ = \sum_{n=0}^{\infty} g_{kn}^+,$$

where g_{kn}^+ are the spherical harmonic of order n.

$$g_{kn}^+ = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma) g_k^+ dS_y,$$

$P_n(\cos\gamma)$ is Legendre polynomial of the n-th order $\cos\gamma$ is an angle formed by the radius-vectors OX and OY

$$\cos\gamma = \frac{1}{|\mathbf{x}||\mathbf{y}|}\sum_{k=1}^{3} x_k y_k$$
$$= \sin\xi\sin\xi_0\cos(\eta-\eta_0) + \cos\xi\cos\xi_0.$$

Substituting the expressions (15) and (16) into (14), taking into account boundary conditions and passing to the limit as $\rho \to R$, for determining the unknown values we obtain the following system of equations:

$$\Omega^+ + 2m_0 h_0 - \rho\frac{\partial}{\partial\rho}\left[(m_0-1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right] = g_1^+,$$
$$A_0 h^+ + \sum_{j=1}^{3} A_j h_j^+ = g_3^+, \quad B_0 h^+ + \sum_{j=1}^{3} B_j h_j^+ = g_4^+, \qquad (18)$$
$$C_0 h^+ + \sum_{j=1}^{3} C_j h_j^+ = g_5^+, \quad h^+ + \sum_{j=1}^{3} h_j^+ = g_2^+.$$

From $(18)_2$ and $(18)_3$ we get

$$h^+(\xi,\eta) =$$
$$\frac{1}{m_0\mu}[\mu_0 g_2^+ + b g_3^+ - \beta g_4^+ - \gamma_0 g_5^+] = G^+(\xi,\eta). \qquad (19)$$

Let us consider the following systeme of quations

$$\sum_{j=1}^{3} A_j h_j^+ = g_3^+ - A_0 G^+ = q_1,$$
$$\sum_{j=1}^{3} B_j h_j^+ = g_4^+ - B_0 G^+ = q_2, \qquad (20)$$
$$\sum_{j=1}^{3} C_j h_j^+ = g_5^+ - C_0 G^+ = q_3,$$

Following Theorem 1 we conclude that the determinant d of system (20) is different from zero and the system (20) is uniquely solvable.

From (20) we find

$$h_1^+ = \frac{1}{d}[q_1(B_2C_3 - B_3C_2) - q_2(A_2C_3 - A_3C_2)$$
$$+ q_3(A_2B_3 - A_3B_2)] = \mathrm{H}_1(\xi,\eta),$$

$$h_2^+ = \frac{1}{d}[-q_1(B_1C_3 - B_3C_1) + q_2(A_1C_3 - A_3C_1)$$
$$- q_3(A_1B_3 - A_3B_1)] = \mathrm{H}_2(\xi,\eta),$$
$$h_3^+ = \frac{1}{d}[q_1(B_1C_2 - B_2C_1) - q_2(A_1C_2 - A_2C_1) \qquad (21)$$
$$+ q_3(A_1B_2 - A_2B_1)] = \mathrm{H}_3(\xi,\eta),$$

$$d = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix} \neq 0,$$

Thus the functions $h,\ \ h_j,$ are known,from (18)we get

$$\Omega^+ = g_1^+ - 2m_0 h_0$$
$$+ \rho\frac{\partial}{\partial\rho}\left[(m_0-1)h_0 + \sum_{j=1}^{3}\frac{h_j}{\lambda_j^2}\right] = G_4^+, \qquad (22)$$

On the other hand from (19),(21)and (22),we get

$$Z_n = G_n^+, \quad Y_n = G_{4n}^+, \quad Y_{jn} = H_{jn},$$

where

$$G_n^+(\xi,\eta) = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma) G^+ dS_y,$$
$$G_{4n}^+(\xi,\eta) = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma) G_4^+ dS_y,$$

$$H_{jn}(\xi,\eta) = \frac{2n+1}{4\pi R^2}\iint_S P_n(\cos\gamma) H_j^+ dS_y.$$

Through inserting the obtained values into (15), we obtain

$$h(\mathbf{x}) = \frac{1}{4\pi R}\iint_S \frac{R^2-\rho^2}{|\mathbf{x}-\mathbf{y}|^3} G^+(y) ds,$$

$$\Omega(\mathbf{x}) = \frac{1}{4\pi R}\iint_S \frac{R^2-\rho^2}{|\mathbf{x}-\mathbf{y}|^3} G_4^+(y) ds,$$

$$h_j = \sum_{n=0}^{\infty} \Phi_n(\lambda_j \rho) H_{jn}(\xi,\eta), \quad j = 1,2,3, \quad \rho < R,$$

$$h_0 = \frac{\rho^2}{2} \sum_{n=0}^{\infty} \frac{1}{3+2n} \left(\frac{\rho}{R}\right)^n G_n^+(\theta,\eta).$$

We assume that the functions f_k^+. $k = 1,...,5,$ satisfy the following conditions on S

$$f_k^+ \in C^5(S), \quad j = 1,2,3,4,5.$$

Under these conditions the resulting series are absolutely and uniformly convergent.

5.Explicit solution of the Problem 2

Following the procedure, quite similarly as above, we can construct a solution of the Problem 2 for a thermoelastic porous space with a spherical cavity

Note that the following identities are holds:

$$\frac{\partial}{\partial n} \operatorname{grad} g = \frac{1}{\rho} \operatorname{grad}\left(\rho \frac{\partial}{\partial \rho} - 1\right) g,$$

$$\frac{\partial}{\partial n} rot\, \mathbf{h} = \frac{1}{\rho} \operatorname{rot}\left[\left(\rho \frac{\partial}{\partial \rho} - 1\right) \mathbf{h},\right]$$

$$\operatorname{div} \mathbf{n} g = \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) g,$$

$$\operatorname{div}[\mathbf{n} \cdot rot \mathbf{u}] = -\frac{\partial}{\partial \rho} \operatorname{div} \mathbf{\Psi}, \quad if \quad \Delta \mathbf{\Psi} = 0,$$

$$\lambda \operatorname{div} \mathbf{u} - b\varphi - \beta p - \gamma_0 \theta = \mu(m_0 - 2)h - 2\mu \sum_{j=1}^{3} h_j,$$

$$\operatorname{div} \frac{\partial \mathbf{u}}{\partial n} = \left(\frac{\partial}{\partial \rho} + \frac{1}{\rho}\right) \operatorname{div} \mathbf{u} - \frac{1}{\rho}\left(\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial n}\right),$$

$$\left(\mathbf{n} \cdot \frac{\partial \mathbf{u}}{\partial n}\right) = \frac{1}{\rho}\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right) - \frac{\partial^2}{\partial \rho^2}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3} \frac{h_j}{\lambda_j^2}\right],$$

$$\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right) = \frac{\partial(\mathbf{x} \cdot \mathbf{\Psi})}{\partial n} - \frac{(\mathbf{x} \cdot \mathbf{\Psi})}{\rho}, \quad (\mathbf{x} \cdot \operatorname{grad}) = \rho \frac{\partial}{\partial \rho}.$$

Taking into account these identities, let us rewrite the stress vector in the following form

$$\mathbf{P}(\partial_x, \mathbf{n})\mathbf{u} = 2\mu \frac{\partial \mathbf{\Psi}}{\partial n}$$
$$-\frac{2\mu}{\rho} \operatorname{grad}\left(\rho \frac{\partial}{\partial \rho} - 1\right)\left[(m_0 - 1)h_0 + \sum_{j=1}^{3} \frac{h_j}{\lambda_j^2}\right]$$
$$+ \mathbf{n}\left[\mu(m_0 - 2)h - 2\mu \sum_{j=1}^{3} h_j\right] + \frac{\mu}{\rho}[\mathbf{x} \cdot rot \mathbf{\Psi}] \quad (23)$$

By direct calculation,from (23) we easily derive

$$(\mathbf{x} \cdot \mathbf{PU}) = 2\mu\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right)$$
$$-2\mu\rho \frac{\partial^2}{\partial \rho^2}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3} \frac{h_j}{\lambda_j^2}\right]$$
$$+ \rho\left[\mu(m_0 - 2)h - 2\mu \sum_{j=1}^{3} h_j\right], \quad (24)$$

$$\operatorname{div}(\mathbf{P}(\partial_x, \mathbf{n})\mathbf{U}) = \frac{\mu\, m_0 h}{\rho} - \frac{2\mu}{\rho^2}\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right)$$
$$+ \frac{2\mu}{\rho} \frac{\partial^2}{\partial \rho^2}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3} \frac{h_j}{\lambda_j^2}\right].$$

For convenience we introduce the following functions:

$$(\mathbf{x} \cdot \mathbf{f})^- = g_1^-, \quad (\operatorname{div} \mathbf{f})^- = g_2^-,$$
$$\left(\frac{\partial \varphi}{\partial n}\right)^- = g_3^-, \quad \left(\frac{\partial p}{\partial n}\right)^- = g_4^-, \quad \left(\frac{\partial \theta}{\partial n}\right)^- = g_5^-.$$

By passing to the limit as $\rho \longrightarrow R$, employing

(24), and (11), we derive the following

system of equations

$$\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right)^- - R\left(\frac{\partial^2}{\partial \rho^2}\left[(m_0 - 1)h_0^- + \sum_{j=1}^{3} \frac{h_j^-}{\lambda_j^2}\right]\right)_{\rho=R}$$
$$+ \frac{R}{2}\left((m_0 - 2)h^- - 2\sum_{j=1}^{3} h_j^-\right)_{\rho=R} = \frac{g_1^-}{2\mu},$$

$$m_0 h - \frac{2}{R}\left(\left(\mathbf{x} \cdot \frac{\partial \mathbf{\Psi}}{\partial n}\right)\right)^-$$
$$+ 2\left(\frac{\partial^2}{\partial \rho^2}\left[(m_0 - 1)h_0 + \sum_{j=1}^{3} \frac{h_j}{\lambda_j^2}\right]\right)_{\rho=R} = \frac{R}{\mu} g_2^-, \quad (25)$$

$$(m_0-1)\left(\frac{\partial h}{\partial n}\right)^- - \sum_{j=1}^{3}\left(\frac{\partial h_j}{\partial n}\right)^- = \frac{R}{2\mu}\left(\frac{\partial}{\partial \rho}\left(g_2+\frac{g_1}{R^2}\right)\right)_{\rho=R},$$

$$A_0\left(\frac{\partial h}{\partial n}\right)^- + \sum_{j=1}^{3} A_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_3^-,$$

$$B_0\left(\frac{\partial h}{\partial n}\right)^- + \sum_{j=1}^{3} B_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_4^-,$$

$$C_0\left(\frac{\partial h}{\partial n}\right)^- + \sum_{j=1}^{3} C_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_5^-.$$

From here we get

$$\left(\frac{\partial h}{\partial n}\right)^- = -\frac{\delta_0}{d_3(\lambda+\mu)}[-\mu_0\frac{R}{2\mu}(g_2^- + \frac{g_1^-}{R^2}) + bg_3^- \qquad (25)_a$$
$$-\beta g_4^- - \gamma_0 g_5^-] = G^-.$$

Let us consider the following system of equations

$$\sum_{j=1}^{3} A_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_3^- - A_0 G^- = q_1,$$
$$\sum_{j=1}^{3} B_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_4^- - B_0 G^- = q_2, \qquad (26)$$
$$\sum_{j=1}^{3} C_j\left(\frac{\partial h_j}{\partial n}\right)^- = g_5^- - C_0 G^- = q_3.$$

Hence, following Theorem 1, we conclude, that the determinant of system (26) is different from zero. On solving the equations (26), similarly as above section, we get

$$\left(\frac{\partial h_1}{\partial n}\right)^- = \frac{1}{d}[q_1(B_2C_3 - B_3C_2) - q_2(A_2C_3 - A_3C_2)$$
$$+ q_3(A_2B_3 - A_3B_2)] = G_1^-,$$

$$\left(\frac{\partial h_2}{\partial n}\right)^- = \frac{1}{d}[-q_1(B_1C_3 - B_3C_1) + q_2(A_1C_3 - A_3C_1)$$
$$- q_3(A_1B_3 - A_3B_1)] = G_2^-,$$

$$\left(\frac{\partial h_3}{\partial n}\right)^- = \frac{1}{d}[q_1(B_1C_2 - B_2C_1) - q_2(A_1C_2 - A_2C_1) \qquad (27)$$
$$+ q_3(A_1B_2 - A_2B_1)] = G_3^-,$$

$$\left(\mathbf{x}\cdot\frac{\partial \mathbf{\Psi}}{\partial n}\right)^- = R\left(\frac{\partial^2}{\partial \rho^2}\left[(m_0-1)h_0^- + \sum_{j=1}^{3}\frac{h_j^-}{\lambda_j^2}\right]\right)_{\rho=R}$$
$$-\frac{R}{2}\left((m_0-2)h^- - 2\sum_{j=1}^{3} h_j^-\right)_{\rho=R} + \frac{g_1^-}{2\mu} = G_4^-,$$

where

$$\left(\mathbf{x}\cdot\frac{\partial \mathbf{\Psi}}{\partial n}\right) = \left(\frac{\partial}{\partial \rho} - \frac{1}{\rho}\right)(\mathbf{x}\cdot\mathbf{\Psi}),$$

$$(\mathbf{x}\cdot\mathbf{\Psi}) = \Omega + 2m_0h_0, \qquad \Delta\Omega = 0.$$

We are looking for a solution of the system (1), under BCs of Problem 2, in the form (11), where the functions h and h_j are sought in the form [26]

$$h = -\sum_{n=0}^{\infty}\frac{R^{n+2}}{(n+1)\rho^{n+1}}Z_n(\xi,\eta), \qquad \rho > R,$$
$$(28)$$
$$h_j = \sum_{n=0}^{\infty}\Phi_n(\lambda_j\rho)Y_{jn}(\xi,\eta), \quad j = 1,2,3,$$

Z_n, and Y_{jn}, $j = 1,2,3,$ are the unknown spherical harmonic of order n

$$\Phi_n(\lambda_j\rho) = \frac{\sqrt{R}H^{(1)}_{n+\frac{1}{2}}(\lambda_j\rho)}{\sqrt{\rho}H^{(1)}_{n+\frac{1}{2}}(\lambda_jR)}, \quad \rho > R,$$

$H^{(1)}_{n+\frac{1}{2}}(\lambda_j\rho)$ is the Hankel function.

Taking into account (28), we can write the particular solutions of equation $\Delta h_0 = h$ in the form

$$h_0 = \frac{\rho^2}{2}\sum_{n=0}^{\infty}\frac{1}{1-2n}\frac{R^{n+2}}{\rho^{n+1}}Z_n(\xi,\eta). \qquad (29)$$

On the other hand, from $(25)_a$, (27) and (28), we get

$$Z_n(\xi,\eta) = G_n^-, \quad \rho > R,$$

$$Y_{jn}(\xi,\eta) = \frac{G_{jn}^-}{\omega_n(\lambda_jR)}, \quad j = 1,2,3,$$

$$\omega_n(\lambda_jR) = \left(\frac{\partial}{\partial \rho}\Phi_n(\lambda_j\rho)\right)_{\rho=R},$$

where G_n^- and G_{jn}^- are the spherical harmonics of order n

$$G^{-}=\sum_{n=0}^{\infty}G_{n}^{-},\quad G_{j}^{-}=\sum_{n=0}^{\infty}G_{jn}^{-},$$

$$G_{n}^{-}=\frac{2n+1}{4\pi R^{2}}\iint_{S}P_{n}(\cos\gamma)G^{-}(\xi_{0},\eta_{0})dS_{y},$$

$$G_{jn}^{-}=\frac{2n+1}{4\pi R^{2}}\iint_{S}P_{n}(\cos\gamma)G_{j}^{-}(\xi_{0},\eta_{0})dS_{y}.$$

Using (29) and (28), we obtain

$$h=-\sum_{n=0}^{\infty}\frac{R^{n+2}}{(n+1)\rho^{n+1}}G_{n}(\xi,\eta),$$

$$h_{j}=\sum_{n=0}^{\infty}\frac{\Phi_{n}(\lambda_{j}\rho)}{\omega(\lambda_{j}R)}G_{jn}^{-}(\xi_{0},\eta_{0}),\quad \rho>R,$$

$$h_{0}=\frac{\rho^{2}}{2}\sum_{n=0}^{\infty}\frac{1}{1-2n}\frac{R^{n+2}}{\rho^{n+1}}G_{n}(\xi,\eta).$$

Assuming the functions h, h_j and h_0 known, from (27), when $\rho=R$, we get

$$\left(\frac{\partial\Omega}{\partial n}-\frac{\Omega}{R}\right)^{-}=G_{4}^{-}-2m_{0}\left(\frac{\partial}{\partial\rho}-\frac{1}{R}\right)h_{0}=G_{5}^{-}.$$

Thus, we have obtained for the Laplace equation

$$\Delta\Omega=0,\quad x\in D^{-},$$

the Robin boundary value problem

$$\left(\frac{\partial\Omega}{\partial n}-\frac{\Omega}{R}\right)^{-}=G_{5}^{-},$$

the solution of which has the form

$$\Omega(\mathbf{x})=\frac{1}{2\pi}\iint_{S}\frac{g(y)}{r(x,y)}ds, \tag{30}$$

where $g(y)$ is a solution of the following Fredholm integral equation of second kind

$$-g(x)+\frac{1}{2\pi}\iint_{S}\frac{\partial}{\partial n_{x}}\frac{1}{r(x,y)}g(y)ds-\frac{1}{2\pi R}\iint_{S}\frac{g(y)}{r(x,y)}ds=G_{5}^{-}, \tag{31}$$

$$r^{2}(x,y)=\sum_{j=1}^{3}(x_{j}-y_{j})^{2}.$$

It is well known that integral equation (31) is always solvable.

Substituting the obtained values into (28) and (11), we get the final form for solution of the considered Problem 2.

We assume that the functions f_k^- satisfy the following conditions on S

$$f_{k}^{-}\in C^{5}(S),\quad j=1,2,3,4,5.$$

Under these conditions the resulting series are absolutely and uniformly convergent. Moreover we assume that the functions $\hat{\mathbf{G}}^{-}$, $\hat{\mathbf{G}}_{k}^{-}$ and its first order derivatives are absolutely integrable and vanishing at infinity functions.

Thus, the considered problems are completely solved.

6.Conclusions

The main results of this work can be formulated as follows:

1. The general solution of the system of equations in the considered theory is presented by means of elementary (harmonic, meta-harmonic and bi-harmonic) functions;

2. Explicit solutions of problems for a sphere and for a space with spherical cavity is presented. The obtained solutions are given by means of the harmonic, bi-harmonic and meta-harmonic functions. For the harmonic functions the Poisson type formulas are obtained. The bi-harmonic and meta-harmonic functions are presented as absolutely and uniformly convergent series.The harmonic function $\Omega(\mathbf{x})$ is given by (30), where the function $g(y)$ is a solution of the Fredholm integral equation of second kind.

Declaration of Conflict of Interests

The author declares that there is no conflict of interest. They have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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