



Solution of the problems of quasi-statics for an elastic body with double porosity

Ivane Tsagareli^{*1}, Bakur Gulua¹

¹I. Vekua Institute of Applied Mathematics of I. Javakhishvili Tbilisi State University, 2, University St., 0186 Tbilisi, Georgia

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Abstract

The construction of solutions in explicit form is especially important from the point of view of its application, since it makes it possible to effectively carry out a quantitative analysis of the problem under study. This paper investigates the processes of deformation of solids in the quasi-static case. Two-dimensional boundary value problems of Dirichlet and Neumann for an elastic body with double porosity are considered. In Using the Laplace transform, these problems are reduced to auxiliary boundary value problems. Special representations of solutions to auxiliary boundary value problems are constructed using elementary functions that allow reducing the original system of equations to equations of a simple structure and facilitate the solution of the original problems. Auxiliary boundary value problems are solved for a specific elastic body - a porous disk. Solutions to these problems are obtained in the form of series. Conditions are provided that ensure the absolute and uniform convergence of these series and the use of the inverse Laplace theorem. It is proved that the inverse transforms provide a solution to the initial problems.

1. Introduction

If the reasons causing the motion of the body change very slowly in time, then the inertial terms in the equations of motion can be neglected and the problem can be regarded as quasi-static. Quasi-static problems of the theory of elasticity and thermoelasticity are well studied (see e.g. Nowacki (1975)[1]). The quasi-static theory for elastic materials with double porosity was proposed by Aifantis and his co-authors [2-5]. This theory, which the authors called the consolidation theory, combines the Barenblatt model [6] for fluid flow in a double porous medium and the Biot model [7] for elastic materials with a single porosity.

In [8,9], the basic equations were obtained in the quasi-static case for liquid-saturated media with double porosity. The fundamental solution of quasi-static equations of the linear of theory elasticity for double porosity solids is constructed and basic properties are established in [10] and [11]. In paper[12] the linear quasi-static theory of thermoelasticity for solids with double porosity is considered. A wide class of the internal and external boundary value problems are formulated. The Green's formulas are obtained. The formulas of integral representations of regular solutions are established, and the uniqueness theorems for classical solutions of the boundary value problems are proved. The history of development of porous body mechanics, are set forth in detail in the monographs [13-15]. Along with theoretical studies of quasi-statics problems, the development of methods for their solution is of great interest.

The topical is the construction of solutions of problems in an explicit form, which is convenient for engineering practice and enables one to perform numerical analysis of the problems under investigation.

This paper investigates the processes of deformation of solids in the quasi-static case. We consider two-dimensional boundary value problems of elasticity for an elastic body with double porosity. In Section 2 the basic equations of the quasi-static of fluid-saturated media with double porosity are given, the basic initial boundary value problems are formulated for an isotropic elastic body with double porosity. Using the Laplace transform, these problems are reduced to auxiliary boundary value problems. In Section 3, we establish Green's identities and prove uniqueness theorems for solutions of auxiliary problems corresponding to the original one. In Section 4 the general representations of solutions of auxiliary boundary value problems are

constructed by means of elementary functions. Examples of the application of these representations are given in Section 5.

Auxiliary boundary value problems are solved for a specific elastic body - a porous disk. Solutions to these problems are obtained in the form of series. Conditions are provided that ensure the absolute and uniform convergence of these series and the use of the inverse Laplace theorem. It is proved that the inverse transforms provide a solution to the initial problems.

2. Formulation of boundary value problems

Let K be a closed curvesurrounding a finite plane region D . In what follows we consider that an isotropic and homogeneous porous elastic solid occupies a region of D . A system of equations of the linear quasi-static theory of elastic materials with double porosity has the form [5]

$$\begin{aligned} &\mu \Delta \mathbf{u}(\mathbf{x}, t) + (\lambda + \mu) \text{grad div} \mathbf{u}(\mathbf{x}, t) - \\ &\text{grad} (\beta_1 p_1(\mathbf{x}, t) + \beta_2 p_2(\mathbf{x}, t)) = 0 \\ &m_1 \Delta p_1(\mathbf{x}, t) - \alpha_1 \partial_t p_1(\mathbf{x}, t) + k(p_2(\mathbf{x}, t) - \\ &p_1(\mathbf{x}, t)) - \beta_1 \text{div} \partial_t \mathbf{u}(\mathbf{x}, t) = 0 \\ &m_2 \Delta p_2(\mathbf{x}, t) - \alpha_2 \partial_t p_2(\mathbf{x}, t) - k(p_2(\mathbf{x}, t) - \\ &p_1(\mathbf{x}, t)) - \beta_2 \text{div} \partial_t \mathbf{u}(\mathbf{x}, t) = 0, \end{aligned} \tag{1}$$

where $\mathbf{x} = (x_1, x_2) \in D$, $t \in T$, $T \equiv [0, \infty)$ is time interval;

$\lambda, \mu, \beta_i, m_i, \alpha_i, k$ are the known elastic and physical

constants (see e.g. [5], [8]), $\mathbf{u}(\mathbf{x}, t) = (u_1, u_2)$ is the displacement

*Corresponding Author: i.tsagareli@yahoo.com,
 (I. Tsagareli Orsid: 0000-0002-8896-5469)

of the point \mathbf{X} ; $p_1(\mathbf{x}, t)$ and $p_2(\mathbf{x}, t)$ are the average pressure values in the neighborhood of \mathbf{X} in the cracks and pores, respectively. It is assumed that $m_i > 0, k > 0, i = 1, 2$.

Let us formulate the following initial boundary value problems.

Find in the domain $D \times T$ a regular solution

$$\mathbf{U}(\mathbf{x}, t) = (\mathbf{u}(\mathbf{x}, t), p_1(\mathbf{x}, t), p_2(\mathbf{x}, t)) \text{ of system (1)}$$

$(\mathbf{U}(\mathbf{x}, t) \in C^1(\bar{D} \times T) \cap C^2(D \times T), \bar{D} = D \cup K)$, that satisfies the initial condition

$$\mathbf{U}(\mathbf{x}, 0) = 0, \tag{2}$$

and, on the boundary D , one of the conditions:

$$\mathbf{u}(\mathbf{z}, t) = \mathbf{f}(\mathbf{z}, t), \quad p_i(\mathbf{z}, t) = f_{i+2}(\mathbf{z}, t), \quad i = 1, 2, \tag{3}$$

in **Problem I**;

$$\mathbf{R}(\partial_z, \mathbf{n})\mathbf{U}(\mathbf{z}, t) = \mathbf{f}(\mathbf{z}, t), \quad \partial_n p_i(\mathbf{z}, t) = f_{i+2}(\mathbf{z}, t), \tag{4}$$

$i = 1, 2$, in **Problem II**,

where $\mathbf{z} = (z_1, z_2) \in K$, $\mathbf{n}(\mathbf{z}) = (n_1(\mathbf{z}), n_2(\mathbf{z}))$ is the external normal to K ; $\mathbf{f} = (f_1, f_2)$, f_1, f_2, f_3, f_4 are the

given functions on K ; $\partial_l = \frac{\partial}{\partial l}$.

$$\mathbf{R}(\partial_x, \mathbf{n})\mathbf{U}(\mathbf{x}, t) = \mathbf{T}(\partial_x, \mathbf{n})\mathbf{u}(\mathbf{x}, t) - \beta_1 \mathbf{n}(\mathbf{x})p_1(\mathbf{x}, t) - \beta_2 \mathbf{n}(\mathbf{x})p_2(\mathbf{x}, t) \tag{5}$$

is the stress vector in the theory of elasticity for materials with double porosity and

$$\mathbf{T}(\partial_x, \mathbf{n})\mathbf{u}(\mathbf{x}, t) = \mu \partial_n \mathbf{u}(\mathbf{x}, t) + \lambda \mathbf{n}(\mathbf{x}) \operatorname{div} \mathbf{u}(\mathbf{x}, t) + \mu \sum_{i=1}^2 n_i(\mathbf{x}) \operatorname{grad} u_i(\mathbf{x}, t)$$

is the stress vector in the classical theory of elasticity.

It is assumed that the functions $\mathbf{U}(\mathbf{x}, t)$ and $f_j(\mathbf{z}, t)$, as well as their derivatives at large t in absolute value, are estimated by the value $Me^{\xi_0 t}$, where $M > 0$ and $\xi_0 \geq 0$ are constants, $\mathbf{x} \in D$, $\mathbf{z} \in K$.

Applying the formal Laplace transform

$$(\tilde{\mathbf{U}}(\mathbf{x}, \tau), \tilde{f}_j(\mathbf{z}, \tau)) = \int_0^\infty e^{-t\tau} (\mathbf{U}(\mathbf{x}, t), f_j(\mathbf{z}, t)) dt, \tag{6}$$

where $\tau = \xi + i\eta$, $\operatorname{Re} \tau = \xi > 0$, $\operatorname{Im} \tau = \eta \in (-\infty; \infty)$,

$\tilde{\mathbf{U}} = (\tilde{\mathbf{u}}, \tilde{p}_1, \tilde{p}_2)$, we can reduce problems I and II to the following auxiliary boundary value problems:

$$\begin{aligned} &\mu \Delta \tilde{u}(\mathbf{x}, \tau) + (\lambda + \mu) \operatorname{grad} \operatorname{div} \tilde{u}(\mathbf{x}, \tau) - \\ &\beta_1 \operatorname{grad} \tilde{p}_1(\mathbf{x}, \tau) - \beta_2 \operatorname{grad} \tilde{p}_2(\mathbf{x}, \tau) = 0 \\ &m_1 \Delta \tilde{p}_1(\mathbf{x}, \tau) - \alpha_1 \tau \tilde{p}_1(\mathbf{x}, \tau) + \\ &k[\tilde{p}_2(\mathbf{x}, \tau) - \tilde{p}_1(\mathbf{x}, \tau)] - \beta_1 \tau \operatorname{div} \tilde{u}(\mathbf{x}, \tau) = 0 \\ &m_2 \Delta \tilde{p}_2(\mathbf{x}, \tau) - \alpha_2 \tau \tilde{p}_2(\mathbf{x}, \tau) - \\ &k[\tilde{p}_2(\mathbf{x}, \tau) - \tilde{p}_1(\mathbf{x}, \tau)] - \beta_2 \tau \operatorname{div} \tilde{u}(\mathbf{x}, \tau) = 0; \end{aligned} \tag{7}$$

$$\tilde{\mathbf{u}}(\mathbf{z}, \tau) = \tilde{\mathbf{f}}(\mathbf{z}, \tau), \quad \tilde{p}_i(\mathbf{z}, \tau) = \tilde{f}_{i+2}(\mathbf{z}, \tau), \quad i = 1, 2, \text{ Problem } I_\tau; \tag{8}$$

$$\mathbf{R}(\partial_z, \mathbf{n})\tilde{\mathbf{U}}(\mathbf{z}, \tau) = \tilde{\mathbf{f}}(\mathbf{z}, \tau),$$

$$\partial_R \tilde{p}_i(\mathbf{z}, \tau) = \tilde{f}_{i+2}(\mathbf{z}, \tau) - \text{Problem } II_\tau. \tag{9}$$

3. General representation of the solution of system (7)

Applying the operator *div* to the first equation of system (7), we obtain a system of equations with respect to the sought value *div* $\tilde{\mathbf{u}}$, \tilde{p}_1 and \tilde{p}_2 . The determinant of this system has the following form

$$\begin{aligned} \det(\Delta) = &\Delta \{ \mu_0 m_1 m_2 \Delta^2 - [\mu_0 (\alpha_2 m_1 \tau + \alpha_1 m_2 \tau + \\ &k m_1 + k m_2) + \beta_1^2 m_2 \tau + \beta_2^2 m_1 \tau] \Delta + [\mu_0 (\alpha_1 \alpha_2 \tau + \\ &k \alpha_1 + k \alpha_2) + 2 \beta_1 \beta_2 k + \beta_1^2 \alpha_2 \tau + \beta_2^2 \alpha_1 \tau + \\ &k \beta_1^2 + k \beta_2^2] \tau \}, \end{aligned}$$

where $\mu_0 = \lambda + 2\mu > 0$. The expression in curly braces with respect to Δ is a square trinomial with complex coefficients, $\mu_0 m_1 m_2 > 0$. We write the determinant $\det(\Delta)$ in the form of a product $\det(\Delta) = \mu_0 m_1 m_2 \Delta (\Delta + \omega_2^2) (\Delta + \omega_3^2)$. (18)

where $(-\omega_j^2)$ are mutually conjugate complex roots that differ from each other, $j = 2, 3$.

Thus, with respect to *div* $\tilde{\mathbf{u}}$, \tilde{p}_1 and \tilde{p}_2 we obtain the new equations

$$\Delta (\Delta + \omega_2^2) (\Delta + \omega_3^2) \begin{pmatrix} \operatorname{div} \tilde{u} \\ \tilde{p}_1 \\ \tilde{p}_2 \end{pmatrix} = 0,$$

where $(-\omega_j^2)$ are mutually conjugate complex roots that differ from each other, $j = 2, 3$. Now, applying the operator

$$\operatorname{rot} = -\partial_2 + \partial_1 \text{ to the first equation of system (7), we obtain } \Delta \varphi_4 = 0,$$

$$\text{where } \operatorname{rot} \tilde{\mathbf{u}} = \varphi_4. \tag{20}$$

$$\text{where } \operatorname{rot} \tilde{\mathbf{u}} = \varphi_4. \tag{19}$$

The general representation of the solution to system (7) can be rewritten as follows [16, 17]

$$\begin{aligned} \tilde{\mathbf{u}} &\equiv (\tilde{u}_1, \tilde{u}_2) = c_1 \text{grad } \varphi_1 + c_2 \text{grad } \varphi_2 + \\ &c_3 \text{grad } \varphi_3 + c_4 \text{rot } \varphi_4 + c_0 \mathbf{\Gamma}, \\ \tilde{p}_1 &= \varepsilon_1 \varphi_1 + \varepsilon_2 \varphi_2 + \varepsilon_3 \varphi_3, \end{aligned} \tag{10}$$

(10)

$$\tilde{p}_2 = \varphi_1 + \varphi_2 + \varphi_3,$$

where $\varphi_l(\mathbf{x}, \tau)$ are scalar metaharmonic functions with a complex argument

$$\Delta \varphi_l = 0, \quad i = 1, 4; \quad (\Delta + \omega_l^2) \varphi_l = 0, \quad l = 2, 3, \tag{11}$$

$\varphi_i(\mathbf{x}, \tau)$ are harmonic functions; $\varphi_4 = \text{rot } \tilde{\mathbf{u}}, \quad \omega_1^2 = 0,$

$\omega_4^2 = 0.$ We will apply the operator *div* to the first of the

representations (10) and compare it to *div* $\tilde{\mathbf{u}}$, calculated by the

formula (7)₂ or (7)₃. Taking into account representations (10),

after simple calculations, we obtain the values of the coefficients c_j

$$c_j = - \left\{ \left[\frac{\mu_0}{\beta_1 \tau} (m_1 \omega_j^2 + \alpha_1 \tau + k) + \beta_1 \right] \varepsilon_j + \frac{\beta_1 \beta_2 \tau - \mu_0 k}{\beta_1 \tau} \right\},$$

$$c_4 = -\mu, \tag{12}$$

where

$$\varepsilon_j = \frac{(\beta_1 \beta_2 \tau - \mu_0 k) \omega_j^2 - k}{(\mu_0 \omega_j^2 + 1)(m_1 \omega_j^2 + \alpha_1 \tau + k) + \beta_1^2 \tau \omega_j^2},$$

$j = 1, 2, 3;$

c_0 is an arbitrary constant, $\mathbf{\Gamma} = (\Gamma_1, \Gamma_2),$

$$\Gamma_1 = x_2, \quad \Gamma_2 = -x_1, \quad \text{div } \mathbf{\Gamma} = 0.$$

By an immediate check we conclude that representations (10) and (12) satisfy equations (7).

4. Construction of solutions of auxiliary boundary value problems for a porous elastic disk

Let region D have the form of a disk bounded by a circumference K with radius R , the center of which coincides with the origin.

Problem I_r . Regular solutions of equations (11) in the circle can be represented as follows [16]:

$$\varphi_1(\mathbf{x}, \tau) = \sum_{m=0}^{\infty} \left(\frac{r}{R} \right)^m (\mathbf{X}_{m1} \cdot \mathbf{v}_m),$$

$$\varphi_2(\mathbf{x}, \tau) = \sum_{m=0}^{\infty} J_m(\omega_2 r) (\mathbf{X}_{m2} \cdot \mathbf{v}_m),$$

$$\varphi_3(\mathbf{x}, \tau) = \sum_{m=0}^{\infty} J_m(\omega_3 r) (\mathbf{X}_{m3} \cdot \mathbf{v}_m), \tag{13}$$

$$\varphi_4(\mathbf{x}, \tau) = \sum_{m=0}^{\infty} \left(\frac{r}{R} \right)^m (\mathbf{X}_{m4} \cdot \mathbf{s}_m), \tag{13}$$

where

$\mathbf{x} = (r, \psi), \quad r^2 = x_1^2 + x_2^2, \quad 0 \leq \psi \leq 2\pi,$ J_m is the Bessel function of a complex variable with integer index m ;

$\mathbf{X}_{ml} = (\mathbf{X}'_{ml}, \mathbf{X}''_{ml})$ is the sought constant vector,

$$\mathbf{v}_m(\psi) = (\cos m\psi, \sin m\psi),$$

$$\mathbf{s}_m(\psi) = (-\sin m\psi, \cos m\psi), \quad l = 1, 2, 3, 4.$$

Let us write the boundary conditions (8) of Problem I_r in the forms of normal and tangent components

$$\tilde{u}_n(\mathbf{z}, \tau) = \tilde{f}_n(\mathbf{z}, \tau), \quad \tilde{u}_s(\mathbf{z}, \tau) = \tilde{f}_s(\mathbf{z}, \tau), \tag{14}$$

$$\tilde{p}_i(\mathbf{z}, \tau) = \tilde{f}_{i+2}(\mathbf{z}, \tau), \quad i = 1, 2,$$

(14)

where $\tilde{f}_n = n_1 \tilde{f}_1 + n_2 \tilde{f}_2, \quad \tilde{f}_s = -n_2 \tilde{f}_1 + n_1 \tilde{f}_2.$

From representation (10) we write

$$\tilde{u}_n(\mathbf{x}, \tau) = \partial_r \sum_{j=1}^3 c_j \varphi_j - \frac{c_4}{r} \partial_\psi \varphi_4,$$

$$\tilde{u}_s(\mathbf{x}, \tau) = \frac{1}{r} \partial_\psi \sum_{j=1}^3 c_j \varphi_j + c_4 \partial_r \varphi_4 - c_0 r. \tag{15}$$

Let us expand the functions \tilde{f}_n, \tilde{f}_s and \tilde{f}_{i+2} into Fourier series and allocate private amounts from each series

$$\tilde{f}_n(\mathbf{z}, \tau) = \sum_{m=0}^{\bar{m}} (\alpha_m(\tau) \cdot \mathbf{v}_m) + M_{\bar{m}n}(\mathbf{z}, \tau),$$

$$\tilde{f}_s(\mathbf{z}, \tau) = \sum_{m=0}^{\bar{m}} (\beta_m(\tau) \cdot \mathbf{s}_m) + M_{\bar{m}s}(\mathbf{z}, \tau), \tag{16}$$

$$\tilde{f}_3(\mathbf{z}, \tau) = \sum_{m=0}^{\bar{m}} (\gamma_m(\tau) \cdot \mathbf{v}_m) + M_{\bar{m}3}(\mathbf{z}, \tau),$$

$$\tilde{f}_4(\mathbf{z}, \tau) = \sum_{m=0}^{\bar{m}} (\delta_m(\tau) \cdot \mathbf{v}_m) + M_{\bar{m}4}(\mathbf{z}, \tau),$$

Where $\alpha_m(\tau) = (\alpha_{m1}, \alpha_{m2}),$

$\beta_m(\tau) = (\beta_{m1}, \beta_{m2}), \quad \gamma_m(\tau) = (\beta_{m1}, \beta_{m2})$ and

$\delta_{mi}(\tau) = (\delta_{mi1}, \delta_{mi2})$ are the Fourier coefficients of the

respective functions \tilde{f}_n, \tilde{f}_s and $\tilde{f}_{i+2}, \quad i = 1, 2; \quad M_{\bar{m}n}(\mathbf{z}, \tau),$

$M_{\bar{m}s}(\mathbf{z}, \tau), \quad M_{\bar{m}3}(\mathbf{z}, \tau)$ and $M_{\bar{m}4}(\mathbf{z}, \tau)$ are cut-off parts

(errors) of the corresponding series, each of which, when

$\tilde{f} \in C^{n+1}(K)$, is estimated as follows:

$$\left| M_{\bar{m}}(\mathbf{z}, \tau) \right| < \frac{c'}{\bar{m}^{n+0.5}}, \quad c' = const > 0, \tag{17}$$

where \bar{m} is a sufficiently large number. Taking into account (10), we substitute representations (13) into (15) and pass to the limit as $r \rightarrow R$. Then, taking (16) into account, from the boundary conditions (14), for each m we obtain a system of linear algebraic equations

$$\begin{aligned} c_1 m \mathbf{X}_{m1} + c_2 R \omega_2 J'_m(\omega_2 R) \mathbf{X}_{m2} + c_3 R \omega_3 J'_m(\omega_3 R) \mathbf{X}_{m3} + \\ c_4 m \mathbf{X}_{m4} = R \alpha_m(\mathbf{z}, \tau), \\ c_1 \mathbf{X}_{m1} + c_2 J_m(\omega_2 R) \mathbf{X}_{m2} + c_3 J_m(\omega_3 R) \mathbf{X}_{m3} + \\ c_4 \mathbf{X}_{m4} = \frac{R}{m} \beta_m(\mathbf{z}, \tau), \end{aligned} \tag{18}$$

$$\begin{aligned} \varepsilon_1 \mathbf{X}_{m1} + \varepsilon_2 J_m(\omega_2 R) \mathbf{X}_{m2} + \varepsilon_3 J_m(\omega_3 R) \mathbf{X}_{m3} = \gamma_m(\mathbf{z}, \tau), \\ \mathbf{X}_{m1} + J_m(\omega_2 R) \mathbf{X}_{m2} + J_m(\omega_3 R) \mathbf{X}_{m3} = \delta_m(\mathbf{z}, \tau), \\ m = 0, 1, 2, \dots, \bar{m}, \end{aligned}$$

where J'_m is the derivative of the function J_m with respect to its argument. For $m = 0$, from the second equation of system (18) we obtain: $c_0 = -\frac{\beta_0}{2}$. For convenience, we introduced the notation

$$X_{04} \equiv c_0. \text{ So, } X_{04} = -\frac{\beta_0}{2}; \quad \beta_0 = -\frac{1}{2\pi} \int_0^{2\pi} \tilde{f}_s(\theta) d\theta.$$

We solve system (18) and substitute the obtained values of the vectors \mathbf{X}_{mj} and \mathbf{X}_{m4} into (13), and thereafter into (10). We obtain the solution of Problem Π_τ .

Problem Π_τ . Let us consider the normal and tangential components of the stress vector (10), we obtain

$$\begin{aligned} \{\mathbf{R}(\partial_x, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{x}, \tau)\}_n = \mu_0 \partial_r \tilde{u}_n(\mathbf{x}, \tau) + \\ \frac{\lambda}{r} \partial_\psi \tilde{u}_s(\mathbf{x}, \tau) - \beta_1 \tilde{p}_1(\mathbf{x}, \tau) - \beta_2 \tilde{p}_2(\mathbf{x}, \tau), \end{aligned} \tag{19}$$

$$\{\mathbf{R}(\partial_x, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{x}, \tau)\}_s = \mu \left[\partial_r \tilde{u}_s(\mathbf{x}, \tau) + \frac{1}{r} \partial_\psi \tilde{u}_n(\mathbf{x}, \tau) \right], \tag{19}$$

where $\tilde{u}_n(\mathbf{x}, \tau)$ and $\tilde{u}_s(\mathbf{x}, \tau)$ are defined by formulas (13). The boundary conditions (9) can be written in the form ($\partial_R = \partial_n$):

$$\begin{aligned} \{\mathbf{R}(\partial_z, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{z}, \tau)\}_n = \tilde{f}_n(\mathbf{z}, \tau), \\ \{\mathbf{R}(\partial_z, \mathbf{n}) \tilde{\mathbf{U}}(\mathbf{z}, \tau)\}_s = \tilde{f}_s(\mathbf{z}, \tau), \\ \partial_R \tilde{p}_1(\mathbf{z}, \tau) = \tilde{f}_3(\mathbf{z}, \tau), \quad \partial_R \tilde{p}_2(\mathbf{z}, \tau) = \tilde{f}_4(\mathbf{z}, \tau), \end{aligned} \tag{20}$$

where \tilde{f}_n , \tilde{f}_s and \tilde{f}_{i+2} are the known functions. It is assumed that they are expandable into Fourier series. Substitution (15) into (19), using (13) and (16) and passing to the limit as $r \rightarrow R$, from

(20) we obtain for each $m = 1, 2, \dots, \bar{m}$ a system of algebraic linear equations

$$A \mathbf{X} = \mathbf{Y}, \tag{21}$$

$$A = \|a_{pq}\|_{4 \times 4}, \quad \mathbf{X} = (X_{m1}, X_{m2}, X_{m3}, X_{m4})^T,$$

$$\mathbf{Y} = (\alpha_m, \beta_m, \gamma_m, \delta_m)^T, \quad p, q = 1, 2, 3, 4,$$

$$a_{11} = \mu_0 m(m-1) c_1 \frac{1}{R^2} - \lambda m c_2 \frac{1}{R^2} - \beta_1 \varepsilon_1 - \beta_2,$$

$$a_{12} = \mu_0 c_2 \omega_2^2 J''_m(\omega_2 R) - (\lambda m c_2 \frac{1}{R^2} - \beta_1 \varepsilon_2 - \beta_2) J_m(\omega_2 R),$$

$$a_{13} = \mu_0 c_3 \omega_3^2 J''_m(\omega_3 R) - (\lambda m c_3 \frac{1}{R^2} -$$

$$\beta_1 \varepsilon_3 - \beta_2) J_m(\omega_3 R),$$

$$a_{14} = \mu_0 c_4 m^2 \frac{1}{R} - \lambda m^2 c_4 \frac{1}{R^2},$$

$$a_{21} = \mu c_1 m(m-1) \frac{1}{R^2} - \mu m^2 c_1 \frac{1}{R^2} - \mu c_0,$$

$$a_{22} = \mu m c_2 \frac{1}{R} \left[-\frac{1}{R} J_m(\omega_2 R) + \omega_2 J'_m(\omega_2 R) \right],$$

$$a_{23} = \mu m c_3 \frac{1}{R} \left[-\frac{1}{R} J_m(\omega_3 R) + \omega_3 J'_m(\omega_3 R) \right],$$

$$a_{24} = \mu m c_4 \frac{m-1}{R}, \quad a_{31} = \frac{m}{R} \varepsilon_1, \quad a_{32} = \varepsilon_2 \omega_2 J'_m(\omega_2 R),$$

$$a_{33} = \varepsilon_3 \omega_3 J'_m(\omega_3 R), \quad a_{34} = 0, \quad a_{41} = \frac{m}{R},$$

$$a_{42} = \omega_2 J'_m(\omega_2 R), \quad a_{43} = \omega_3 J'_m(\omega_3 R), \quad a_{44} = 0.$$

Superscript "T" denotes transposition.

For $m = 0$, from the second equation of system (21) we obtain:

$$-\mu c_0 = \frac{\beta_0}{2}. \text{ Let us introduce the notation } X_{04} = c_0. \text{ So,}$$

$$X_{04} = -\frac{\beta_0}{2\mu}.$$

Substituting the solution of system (21) into (13), and then into (10), we thus find the solutions $\tilde{\mathbf{u}}(\mathbf{x}, \tau)$, $\tilde{p}_1(\mathbf{x}, \tau)$ and $\tilde{p}_2(\mathbf{x}, \tau)$ of Problem Π_τ .

It follows from the uniqueness of the solutions of the problems [12] under consideration that for each m the determinants of systems (18) and (21) are different from zero. Taking into account (17), we note that for large values of m the $M_{\bar{m}n}(\mathbf{z}, \tau)$,

$M_{\bar{m}s}(\mathbf{z}, \tau)$, $M_{\bar{m}3}(\mathbf{z}, \tau)$ and $M_{\bar{m}4}(\mathbf{z}, \tau)$ errors decrease. For each value $m = \bar{m} + 1, \bar{m} + 2, \dots$, the solution of the system of corresponding homogeneous equations will be equal to zero.

By analyzing the general form of solutions \mathbf{X}_{ml} ($l = 1, 2, 3, 4$) of systems (18) and (21)

we can establish that the conditions $f_l(\mathbf{z}, t) \in C^4(0 \leq t < \infty)$ give the following estimate

$$\left| \tilde{\mathbf{U}}(\mathbf{x}, \tau) \right| \leq \frac{c'}{|\tau|^4}, \quad c' = \text{const} \quad (22)$$

for all τ uniformly for each \mathbf{x} , $\mathbf{x} \in \bar{D}$; $l = 1, 2, 3, 4$.

Under condition (22), the Laplace inversion theorem holds (see e.g.

[17]). The vector $\tilde{\mathbf{U}}(\mathbf{x}, \tau)$ is the image, and the original is determined by the integral

$$\mathbf{U}(\mathbf{x}, t) = \frac{1}{2\pi i} \int_{\xi - i\infty}^{\xi + i\infty} e^{t\tau} \tilde{\mathbf{U}}(\mathbf{x}, \tau) d\tau, \quad (23)$$

where $\text{Re } \tau = \xi > \xi_0 \geq 0$, ξ_0 is the growth exponent of the original. Now, using this formula, we can obtain solutions to the original problems I and II. Using condition (22), we can conclude

that the conditions $f_l(\mathbf{z}, t) \in C^4(0 \leq t < \infty)$ and

$f_l(\mathbf{z}, t) \in C^2(K)$ ensure the existence, as well as absolute and uniform convergence: with respect to t originals

$\frac{\partial^l \mathbf{U}}{\partial t^l} \quad (l = 0, 1)$, and with respect to the index m of functions

\mathbf{U} and their derivatives with respect to $z \in K$.

5. Conclusions

1) The main goal of this work is to present explicit solutions of boundary value problems of the linear quasi-static theory of elasticity for solids with double porosity.

2) Special representations are constructed for the general solution of the equations obtained by the Laplace transform. They are expressed in the forms of harmonic and metaharmonic functions, the properties of which are well known in mathematical physics. The solutions of these problems are written explicitly in the form of absolutely and uniformly converging series.

3) The conditions are written under which the inverse Laplace transforms exist and yield solutions of the initial problems.

4) The application of the considered method enables us to investigate a wide class of problems for systems of equations in theories of elasticity and thermoelasticity for materials with double porosity; build explicit solutions of basic boundary value problems not only for a circle, but also for a ring, a plane with a circular hole, etc.

Conflict of Interest

The authors declare that there is no conflict of interest. They have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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